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A critical-amplitude relation for one-dimensional quantum transitions and determination of the exponent η

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Abstract. A new critical-amplitude relation which holds for one-dimensional quantum ground-state transitions is presented. The relation yields an estimate of the sound velocity which appears in the conformal field theory as an *a priori* unknown parameter. With the use of this relation, the exponent η can be explicitly determined from only the scaling fit of the off-critical energy gap. The relation is confirmed in the transverse Ising model, the $S = \frac{1}{2}$ anisotropic XY model, and the three-state Potts model.

1. Introduction

Since the application of the conformal field theory [1–3] to the finite-size scaling [4], it has become an important backup to numerical analyses on phase transitions of two-dimensional systems. One of the remarkable results is the fact [4] that the correlation length $\xi_\parallel$ of a strip at the critical point is related to Fisher's exponent η [5] in the form

$$\xi_\parallel^{-1} \simeq \pi\eta/L \qquad \text{for} \quad L \gg 1 \tag{1}$$

where L denotes the width of the strip.

The theory may be applicable to the ground-state transition in one dimension. It is because a one-dimensional quantum system of size L at zero temperature is mapped to an infinite strip of the relevant two-dimensional classical system. The correlation length $\xi_\parallel$ of the classical system is then translated to that of the quantum system in the imaginary-time direction, ξ_τ, or equivalently [6], to the reciprocal of the energy gap $\Delta E(L)$ above the ground state.

However, we cannot directly estimate the exponent η in the quantum case, because the relation (1) does not hold as it is; it should be modified to include a parameter called the 'sound velocity' v_s [7]

$$\xi_\tau^{-1} = \Delta E(L) \simeq v_s\pi\eta/L \qquad \text{for} \quad L \gg 1 \tag{2}$$

at the critical point. The sound velocity depends on the overall factor of the Hamiltonian, and is not known *a priori*.

In the present paper we introduce a new critical-amplitude relation which emerges in scaling of the energy gap of conformally invariant systems. A ratio of two coefficients in the scaling function equals the sound velocity v_s. Thus we can eliminate v_s from (2), and obtain a universal critical-amplitude ratio which gives the exponent η. (For a review on critical-amplitude ratios, see [8].)

We can utilize this amplitude relation in order to estimate the sound velocity v_s and the exponent η numerically. We would like to stress here that the present method of the estimation is based only on the information of the energy gap between the ground state and the first excited state. Once we calculate the first energy gap for various sizes and parameters, its scaling fit alone yields the critical point, the exponent ν and the exponent η in this order. Note that these three estimates complete the information on the critical properties; we can obtain the other critical exponents by using scaling relations.

Some methods of estimating v_s and η have been proposed so far. One is to construct the so-called virtual-space transfer matrix [9]. Consider an infinite quantum system at finite temperatures. It can be mapped to a strip which is finite in the imaginary-time direction and is infinite in the space direction. The virtual-space transfer matrix transfers states in the space direction. The conformal field theory also predicts that the correlation length in the space direction is given by

$$\xi_x^{-1} = \frac{\pi\eta}{v_s} T \qquad \text{for} \quad T \sim 0 \tag{3}$$

near the critical point. Multiplication of the coefficients in (2) and (3) yields an estimate of η [10].

Another method of estimating the sound velocity v_s is to calculate higher excited states. The conformal field theory also predicts that there is a series of energy levels which has the structure

$$\Delta E_n(L) = \frac{2\pi v_s}{L}\left(\frac{\eta}{2} + n\right) \qquad \text{for} \quad L \gg 1 \tag{4}$$

at the critical point with $\{n\}$ positive integers. If we can somehow find this so-called conformal-tower structure, the sound velocity v_s is estimated from $\Delta E_1 - \Delta E_0$ [11].

These previous methods need calculations *additional* to the estimation of the first energy gap. This is because these methods utilize data only at the critical point, off-critical data being discarded. In contrast, the present method adopts only data of the first energy gap, but those *off* the critical point.

In section 2 we present our main results of the paper, namely the scaling form of the energy gap and the critical-amplitude relation. We derive the relation phenomenologically in section 3 and microscopically in section 4. We confirm the relation for three spin models: analytically in sections 5 and 6, and numerically in section 7.

2. The scaling form of the energy gap

Here we first present the asymptotic form of the scaling function of the first energy gap, and introduce the critical-amplitude relation.

Let us denote the distance from the critical point by ε. We define its sign so that we may have the disordered phase, or the unique ground state for $\varepsilon > 0$. In the region $\varepsilon < 0$ we may have the ordered phase, or degenerate ground states.

We derive the following form of the scaling function:

$$L\Delta E(L) \simeq \begin{cases} Ay + D_+ \mathrm{e}^{-Cy} & \text{for} \quad \varepsilon > 0 \\ B & \text{for} \quad \varepsilon = 0 \\ D_- \mathrm{e}^{-Cy} & \text{for} \quad \varepsilon < 0 \end{cases} \tag{5}$$

where y denotes the scaling variable

$$y \equiv L|\varepsilon|^{\nu}. \tag{6}$$

(Typical examples can be found in figures 3 and 5(*b*), and in [12].)

The coefficients A, B and C are appropriate constants. The prefactors $D_\pm$ of the exponential terms are some moderate functions of y. (We find the behaviour $D_\pm(y) \sim \sqrt{y}$

in sections 4–7 repeatedly. Here we do not investigate further whether this behaviour of the prefactors is universal.)

We can understand the above scaling form (5) roughly as follows.

The term Ay in the first line is due to the ε dependence of the infinite-system energy gap resulting from the energy-operator perturbation

$$\Delta E(\infty) = \xi_\tau^{-1} \simeq A\varepsilon^\nu \tag{7}$$

in the disordered phase, $\varepsilon > 0$. This term does not exist in the third line, since we have $\Delta E(\infty) = 0$ in the ordered phase, $\varepsilon < 0$.

The exponential terms in the first and the third lines of (5) express the asymptotic behaviour of the finite-size energy gap

$$\Delta E(L) - \Delta E(\infty) \sim \exp(-L/\xi_x) \qquad \text{for} \quad L \gg \xi_x \gg 1 \tag{8}$$

where the correlation length has the following singularity:

$$\xi_x^{-1} \simeq C|\varepsilon|^\nu . \tag{9}$$

We derive the expression (8) in section 4.

The exponent in (7) and (9) may be different from ν if we consider perturbation other than the energy-operator one; for example, it should be ν/Δ for the spin-operator perturbation, namely the magnetic field. In that case we define the scaling variable by $y \equiv L|\varepsilon|^{\nu/\Delta}$ instead of $y \equiv L|\varepsilon|^\nu$; the arguments in the present paper remain intact except this redefinition.

The main result in this paper is the following critical-amplitude relation:

$$A/C = v_{\rm s} . \tag{10}$$

The coefficient B, on the other hand, is given by the prediction $\Delta E \simeq v_{\rm s}\pi\eta/L$, namely

$$B = v_{\rm s}\pi\eta . \tag{11}$$

Using the relation (10) we can eliminate the sound velocity from (11); thus we have the universal amplitude ratio

$$BC/A = \pi\eta . \tag{12}$$

Utilizing the scaling form (5) and the above amplitude relation, we propose a new method of estimating the exponent η. First, the L dependence of the scaled energy gap $L\Delta E(L)$ vanishes at the critical point, $\varepsilon = 0$; see figure 5(a) in section 7, for example. Thus the crossing point of the plots of $L\Delta E(L)$ against ε for various values of L yields a critical-point estimate [13, 14]. This is the so-called phenomenological renormalization group. Next, we carry out the scaling fit by scaling ε in the form $y = L|\varepsilon|^\nu$ as in figures 3 and 5(b). This yields an estimate of ν. We can also employ the analysis based on the β function [15, 16] in order to estimate ν. Finally, the amplitude ratio $BC/A = \pi\eta$ introduced here gives an estimate of η.

3. Phenomenological derivation

In the present and the next sections we describe derivation of the amplitude relation $A/C = v_{\rm s}$ in two ways. First, let us discuss it phenomenologically in this section.

First of all, the sound velocity $v_{\rm s}$ appears in the formulation of the conformal field theory of quantum systems as follows.

It is necessary to the applicability of the conformal field theory to quantum systems that the dispersion relation of the relevant elementary excitations is linear near a gapless point k_0, namely

$$\omega(k) \simeq \pm v_s(k - k_0) \qquad \text{for} \quad \varepsilon = 0 \quad \text{and} \quad k \simeq k_0 \,. \tag{13}$$

Then we can describe the low-energy and long-range property of the quantum critical system by means of the two-dimensional massless field theory. The real-space axis x and the imaginary-time axis τ are interchangeable except for a conversion factor given by the 'light cone' $x = v_s\tau$. Thus the theory of a one-dimensional system at the quantum critical point is translated to the conformal field theory of an *anisotropic* two-dimensional system at its thermal critical point.

Strictly speaking, the applicability of the conformal field theory holds only at the critical point; hence the theory involves only the exponent η and the sound velocity v_s. However, we naturally expect that the physics in the critical region around the transition point is also equivalent to that of the anisotropic two-dimensional system.

On account of this expectation, we assume

$$\xi_x \simeq v_s\xi_\tau \qquad \text{for} \quad \varepsilon \simeq 0 \tag{14}$$

see figure 1. The correlation stretches anisotropically near the critical point. The anisotropy of the spacetime may be specified by the sound velocity v_s, and hence the above assumption follows. The assumption (14) relates the off-critical quantities ξ_x and ξ_τ to the critical-point quantity v_s; this is an essential point of the present paper.

Since we define the critical amplitudes A and C in the relations $\xi_\tau^{-1} \simeq A\varepsilon^\nu$ and $\xi_x^{-1} \simeq C\varepsilon^\nu$, the assumption $\xi_x \simeq v_s\xi_\tau$ is followed by the relation $A/C = v_s$.

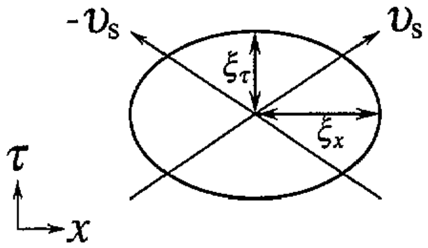

Figure 1. The ellipse indicates a constant-correlation contour. The anisotropy is due to the sound velocity v_s, which is not unity in general.

4. Microscopic derivation

In this section we derive the asymptotic forms (7)–(9) with the relation $A/C = v_s$ more microscopically.

Suppose that we effectively diagonalize the quantum Hamiltonian in terms of the relevant elementary excitations $\{\xi_k^\dagger, \xi_k\}$ in the momentum space. We may write the Hamiltonian of a finite system in the following form:

$$\mathcal{H} = \sum_k \omega(k) \left(\xi_k^\dagger \xi_k - E_0\right) + \text{irrelevant interactions} \,. \tag{15}$$

(This should be considered as a definition of the elementary excitation rather than an assumption of the form of the Hamiltonian. It is unnecessary that the quasi-particles $\{\xi_k^\dagger, \xi_k\}$ are Fermions; they may be bosons [17] or para-Fermions [18].) The indices $\{k\}$ denote the L reciprocal-lattice points; they are arranged at intervals of $2\pi/L$

$$k = \frac{\pi}{L} n \tag{16}$$

with $\{n\}$ even or odd integers. Their parity depends on whether we choose the ground state or the excited state. The difference in the parity results in the finite-size gap in (8) as shown below.

The interactions in the above Hamiltonian (15) is irrelevant perturbation by definition; the elementary excitations is a free massless particle at the relevant fixed point, since the system has conformal symmetry. As the fixed point controls the universality, we may discuss critical properties leaving out the interaction term. Hence the ground-state energy near the critical point may be given only by the leading term

$$E_g(L) \simeq -E_0 \sum_k \omega(k) . \tag{17}$$

(It was proved [19] for the Tomonaga–Luttinger liquid [17, 20, 21] that the interaction does not break the criticality. It was also shown that the nonlinear σ models with the Wess–Zumino term can be described by free para-Fermions [18].)

Allowing for the critical dispersion relation $\omega(k) = \pm v_s(k - k_0)$, we assume the off-critical one in the form

$$\omega(k) \simeq \sqrt{m^2 + v_s^2(k - k_0)^2} \qquad \text{for} \quad k \simeq k_0 \tag{18}$$

with the mass gap

$$m \simeq A\varepsilon^{\nu} . \tag{19}$$

The form (18) naturally appears when the two modes $\omega = v_s(k - k_0)$ and $\omega = -v_s(k - k_0)$ cross with the off-diagonal transition element m; see [22], for example. We can explicitly show that the conditions (15)–(19) are satisfied in the two solvable models in sections 5 and 6.

On one hand, the infinite-system energy gap $\Delta E(\infty)$ in the disordered phase originates in the excitation from the ground state $|0\rangle$ to the state $\xi_{k_0}^{\dagger}|0\rangle$; it is given by the mass gap m in (19), or

$$\Delta E(\infty) = m \simeq A\varepsilon^{\nu} . \tag{20}$$

We thus have the term Ay in the scaling form (5).

On the other hand, the finite-size energy gap $\Delta E(L)$ is estimated as follows. (The estimation is based on the technique developed for the transverse Ising model [16].)

Since the interval of the indices $\{k\}$ is $2\pi/L$, the summation in (17) can be written in terms of an integral of the form

$$E_g(L) = -\frac{LE_0}{2\pi} \int_{-\pi}^{\pi} \delta_L(k + \phi_0)\omega(k)\,\mathrm{d}k \tag{21}$$

where

$$\delta_L(k) \equiv \sum_{l \in Z} \mathrm{e}^{\mathrm{i}kLl} = 1 + \sum_{l=1}^{\infty} \left(\mathrm{e}^{\mathrm{i}kLl} + \mathrm{e}^{-\mathrm{i}kLl}\right) \tag{22}$$

which is the Poisson summation formula. The constant ϕ_0 in (21) is either 0 or π/L, depending on the parity of $\{n\}$ in (16).

The unity in the right-hand side of (22) corresponds to the energy in the thermodynamic limit,

$$E_g(\infty) = -\frac{LE_0}{2\pi} \int_{-\pi}^{\pi} \omega(k)\,\mathrm{d}k. \tag{23}$$

As will be self-evident below, the leading finite-size correction is given by the term $l = 1$ in (22). Therefore the constant ϕ_0 in (21), 0 or π/L, results in the sign of the correction

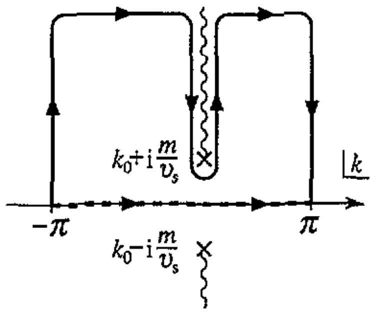

Figure 2. The integration path in (24) is modified from the broken line to the full curve to yield the estimate (25). The crosses are the branch points, and the wavy lines are the branch cuts.

term; the ground state and the excited state approach the thermodynamic limit (23) from below and above, respectively. The finite-size energy gap is thus estimated as

$$\Delta E(L) = \left|E_\mathrm{g}(L;\phi=0) - E_\mathrm{g}(L;\phi=\pi/L)\right| \simeq \frac{2LE_0}{\pi}\left|\mathrm{Re}\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}kL}\omega(k)\,\mathrm{d}k\right| \qquad \text{for} \quad L \gg 1 \tag{24}$$

except the term $A\varepsilon^\nu$ in the disordered phase.

The only singularity of $\omega(k)$ is given by the two branch points $k = k_0 \pm \mathrm{i}m/v_\mathrm{s}$ in the complex plane, as can be seen in (18). We hence modify the integration path as shown in figure 2. The integrals along the paths $\mathrm{Re}\,k = \pm\pi/2$ cancel each other out owing to the periodicity $\omega(k+2\pi) = \omega(k)$. The integrals along the paths $\mathrm{Im}\,k \gg 1$ are exponentially small because of the factor $\mathrm{e}^{\mathrm{i}kL}$. We finally obtain

$$\Delta E(L) \simeq |\cos k_0 L|\,\frac{4v_\mathrm{s}E_0L}{\pi}\exp\left(-\frac{Lm}{v_\mathrm{s}}\right)\int_0^\infty \mathrm{e}^{-\kappa L}\sqrt{\kappa\left(\kappa+\frac{2m}{v_\mathrm{s}}\right)}\,\mathrm{d}\kappa \simeq |\cos k_0 L|\,E_0\sqrt{\frac{8v_\mathrm{s}m}{\pi L}}\exp\left(-\frac{Lm}{v_\mathrm{s}}\right) \tag{25}$$

for $L \gg v_\mathrm{s}/m \gg 1$, except the term $\Delta E(\infty) \simeq A\varepsilon^\nu$ in the disordered phase.

Thus we obtain the finite-size energy gap in the form (8), or the exponential terms in the scaling form (5). Comparing the above expression (25) with the form (8), we have

$$\xi_x^{-1} \simeq m/v_\mathrm{s} \simeq \xi_\tau^{-1}/v_\mathrm{s}\,. \tag{26}$$

Using the relation $m \simeq A\varepsilon^\nu$ and the definition $\xi_x^{-1} \simeq C\varepsilon^\nu$, we again arrive at the relation $A/C = v_\mathrm{s}$.

5. The transverse Ising model

Let us show, in sections 5 and 4, analytic confirmation of the amplitude relation in two exactly solvable examples, namely the transverse Ising model and the $S = \frac{1}{2}$ anisotropic XY model. These two models, after diagonalization, reduce to the form (15) without the interaction term, namely to the free-Fermion model.

First, in this section we consider the transverse Ising model under the periodic boundary condition

$$\mathcal{H} = -\sum_{i=1}^{L}\left(J\sigma_i^x\sigma_{i+1}^x + \Gamma\sigma_i^z\right) \tag{27}$$

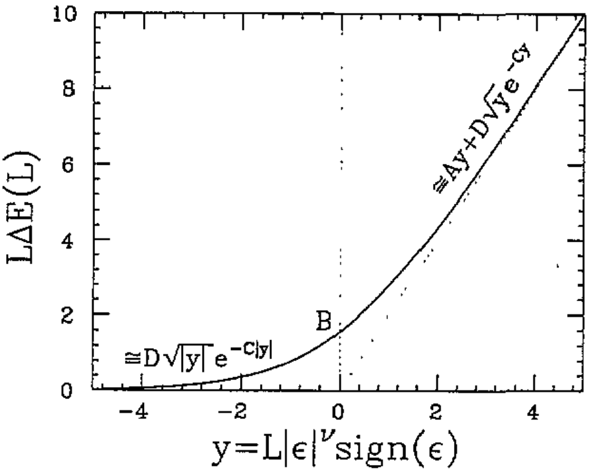

Figure 3. The scaling function of the energy gap for the transverse Ising model (27). The dotted line $L\Delta E = Ay$ is the asymptote in the limit $y \to \infty$. The data were calculated for $J = 1$ and $L = 10^4$. The parameters are $\nu = 1$, $A = 2$, $B = \pi/2$, $C = 1$ and $D = \sqrt{8/\pi}$.

where $\vec{\sigma}$ are the Pauli operators. We can argue [23] that the model is mapped to the anisotropic limit of the two-dimensional Ising model, using the Trotter formula. The two-dimensional Ising model at its critical point was identified with the critical model of the central charge $c = \frac{1}{2}$ [1–3].

We can solve the transverse Ising model exactly by means of the Jordan–Wigner transformation [24, 25]. The critical point is given by $\varepsilon = 0$, where $\varepsilon \equiv \Gamma - J$. In the region $\varepsilon > 0$ the ground state of the infinite system is unique; this is the disordered phase. In the region $\varepsilon < 0$, on the other hand, the two ground states are degenerate, and hence the ordered phase.

The dispersion relation is obtained [24, 25] in the form

$$\omega(k) = \sqrt{m^2 + 4v_s^2 \sin^2 \frac{k - k_0}{2}} \tag{28}$$

with the mass gap $m \equiv 2\varepsilon$, the sound velocity $v_s \equiv 2\sqrt{J\Gamma}$, and the massless point $k_0 \equiv \pi$.

The scaling function of the energy gap is shown in figure 3. The infinite-system energy gap between the ground state $|0\rangle$ and the excited state $\xi_\pi^\dagger|0\rangle$ is given by $\Delta E(\infty) = m = A\varepsilon^\nu$ for $\varepsilon > 0$ with $A = 2$ and $\nu = 1$. The finite-size gap in the critical region is estimated [16]† in the same form as (25) with $E_0 = \frac{1}{2}$, and hence we have $C = 2/v_s$. These coefficients A and C satisfy the relation $A/C = v_s$.

The finite-size energy gap at the critical point is exactly obtained [16] as

$$\begin{aligned}
\Delta E(L) &= v_s\Bigg(\sum_{n=\text{even}} - \sum_{n=\text{odd}}\Bigg)\cos\frac{\pi n}{2L} \\
&= \frac{v_s}{2}\Bigg(\sum_{n=\text{even}} - \sum_{n=\text{odd}}\Bigg)\left(e^{i\pi n/2L} + e^{-i\pi n/2L}\right) \\
&= v_s\left[\cot(\pi/2L) - \operatorname{cosec}(\pi/2L)\right] \\
&= v_s \tan(\pi/4L) \qquad \text{for} \quad \varepsilon = 0
\end{aligned} \tag{29}$$

† In [16], we found errors which were rather typographical themselves but devastating to our results. In equations (3.11) and (4.13) of the reference, the arguments of the exponentials should be doubled. The correct expressions can be easily derived from (A1.16).

where we sum up the geometrical series to move from the second line to the third. This gives the coefficient B as $B = v_s\pi/4$. Thus we can obtain $\eta = \frac{1}{4}$ using the formula $BC/A = \pi\eta$.

6. The $S = \frac{1}{2}$ anisotropic XY model

Next, in this section we show the confirmation of the amplitude relation in the $S = \frac{1}{2}$ anisotropic XY model

$$\mathcal{H} = -\sum_{i=1}^{L} \left(J_x \sigma_i^x \sigma_{i+1}^x + J_y \sigma_i^y \sigma_{i+1}^y \right) . \tag{30}$$

This may be of the universality class of the two-dimensional plane-rotator model, which was identified with the model of the central charge $c = 1$ [3, 26].

The $S = \frac{1}{2}$ XY model is solvable by means of the Jordan–Wigner transformation [24] as well as the transverse Ising model. The isotropic point $J_x = J_y$ is the critical point; hence we define $\varepsilon \equiv J_y - J_x$. The energy spectrum is rather different from the case of the transverse Ising model. Both of the regions $\varepsilon > 0$ and $\varepsilon < 0$ are ordered phases; see figure 4.

The dispersion relation is given [24] by

$$\omega(k) = \sqrt{m^2 + v_s^2 \cos^2 k} \tag{31}$$

with the mass gap $m \equiv 2\varepsilon$ and the sound velocity $v_s \equiv 4\sqrt{J_x J_y}$. There are two massless points, namely $k_0 = \pm\pi/2$. The elementary excitation with the mass m can be regarded as a kink.

The infinite-system energy gap between $|0\rangle$ and $\xi_{\pi/2}^\dagger|0\rangle$ is estimated as $\Delta E(\infty) = m = A\varepsilon^\nu$ with $A = 2$ and $\nu = 1$ for $\varepsilon \neq 0$. Though the finite-size gap becomes twice as large as

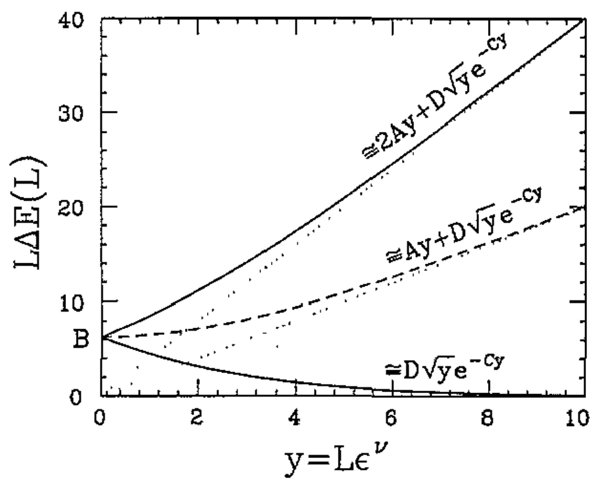

Figure 4. The scaling function of the energy gap for the anisotropic XY model (30). The spectrum is symmetric with respect to $y = 0$. The broken curve indicates the energy gap between the ground state of the periodic chain and that of the antiperiodic chain. The dotted lines $L\Delta E = Ay$ and $L\Delta E = 2Ay$ are the asymptotes. The data were calculated for $J_x = 1$ and $L = 10^4$. The parameters are $\nu = 1$, $A = 2$, $B = 2\pi$, $C = \frac{1}{2}$ and $D = 8/\sqrt{\pi}$.

in the unique-massless-point case (25), the coefficient C does not change from $C = 2/v_s$. We thus confirm the relation $A/C = v_s$ again.

We can calculate the exact finite-size energy gap at the critical point similarly to the case of the transverse Ising model, (29). We obtain

$$\begin{aligned}\Delta E(L) &= \frac{v_s}{2}\left(\sum_{n=\text{even}} - \sum_{n=\text{odd}}\right)\cos\frac{\pi n}{L} \\ &= v_s\left[\cot(\pi/L) - \operatorname{cosec}(\pi/L)\right] \\ &= v_s \tan(\pi/2L) \qquad \text{for} \quad \varepsilon = 0\,. \end{aligned} \tag{32}$$

We thus have $B = v_s\pi/2$. We obtain the exponent $\eta = \frac{1}{2}$ by using the formula $BC/A = \pi\eta$.

Now we have to comment on the definition of the energy gap. As is shown in figure 4, the energy gap of the XY magnet under the periodic boundary condition is twice the mass gap: $\Delta E_{\text{mag}}(\infty) = 2m$. This is because the kinks of the magnet are always excited in pairs. If we employ this definition of the energy gap, we have $A_{\text{mag}} = 4$. Then the relation $A/C = v_s$ appears to be violated. On the contrary, excitation of a single kink is quite possible if the system is viewed as the Fermion system transformed from the magnetic system. The contradiction comes from the fact that the Jordan–Wigner transformation complicates the boundary condition.

As we showed microscopically in section 4, the relation $A/C = v_s$ holds for *each* massless point. It is consistent in our theory to define the energy gap $\Delta E(\infty)$ as the mass of a single elementary excitation. Even in the magnet representation, we can excite a single kink by changing the boundary condition to the antiperiodic one. Then we have the energy gap $\Delta E(\infty) = A\varepsilon^{\nu}$ with $A = 2$. Actually, this gap equals the reciprocal of the correlation length of the 'kink operator'

$$\left\langle O_{\text{kink}}(0)\, O^{\dagger}_{\text{kink}}(\tau)\right\rangle \sim \exp(-\tau \Delta E_{\text{kink}}) \qquad \text{for} \quad \tau \gg 1 \tag{33}$$

where

$$O_{\text{kink}} = \xi_{\pi/2} \sim \left(\prod_{i=-\infty}^{-1} \sigma_i^z\right)\sigma_0^x\,. \tag{34}$$

The kink operator is local in the Fermion representation, though it is non-local in the magnet representation. The phenomenological argument in section 3 holds in the sense (33).

In other words, we can detect how many elementary excitations constitute the energy gap of an infinite system. If a naive estimate of v_s by the present method differs from other estimates by an integer factor, the integer is the number of the elementary excitations. This may be used to confirm the argument on the $S = \frac{1}{2}$ antiferromagnetic Heisenberg chain by Faddeev and Takhtajan [27].

7. The three-state Potts model

Now we confirm the relation $A/C = v_s$ numerically in the quantum version of the three-state Potts model. In contrast to the above $S = \frac{1}{2}$ models, the one-dimensional quantum three-state Potts model is not a free-Fermion model presumably. Thus we can show that the interaction term in (15) is actually irrelevant to the present argument.

We can write in the following form the quantum Hamiltonian corresponding to the three-state Potts model [28]

$$\mathcal{H} = -\sum_{i=1}^{L}\left[\frac{J}{2}\left(R_i^+ R_{i+1}^- + R_i^- R_{i+1}^+\right) + \Gamma\cos\left(\frac{2\pi}{3}L_i\right)\right]. \tag{35}$$

Here the operators $R_i^{\pm}$ and L_i act on a spin state at the site i, $\{|S_i\rangle\} = \{|0\rangle, |1\rangle, |2\rangle\}$, having the following matrix elements, respectively:

$$R_i^+ = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad R_i^- = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad L_i = \begin{pmatrix} 0 & & \\ & 1 & \\ & & 2 \end{pmatrix} . \tag{36}$$

After a unitary transformation we have

$$\mathcal{H} = -\sum_{i=1}^{L} \left\{ J \cos\left[\frac{2\pi}{3}(L_i - L_{i+1})\right] + \frac{\Gamma}{2}\left(R_i^+ + R_i^-\right) \right\} . \tag{37}$$

We can explicitly map this Hamiltonian to the two-dimensional three-state Potts model, using the same technique as in the transverse Ising model [23]. The two-dimensional three-state Potts model was identified with the model of the central charge $c = \frac{4}{5}$ [3, 29, 30].

It has already been shown [31] that analyses of the first energy gap of the present model yielded very precise estimates of the critical point and the exponent ν, namely

$$J_c/\Gamma = 1.000\,00 \pm 0.000\,05 \qquad \text{and} \qquad 1/\nu = 1.2000 \pm 0.0005 . \tag{38}$$

These estimates agree quite well with the predictions $J_c/\Gamma = 1$ and $\nu = \frac{5}{6}$, which are based on the duality relation and the $c = \frac{4}{5}$ conformal field theory. We hence do not repeat the same analyses here; we just assume $\varepsilon = \Gamma - J$ and $\nu = \frac{5}{6}$ in the following, and estimate the sound velocity v_s and the exponent η.

We present our final estimates before mentioning details of our analysis.

We estimated the coefficients A and C as

$$A = 1.600\,08 \pm 0.000\,02 \tag{39}$$

and

$$C = 1.243 \pm 0.005 . \tag{40}$$

We hence estimated the sound velocity v_s as

$$v_s = 1.287 \pm 0.005 . \tag{41}$$

This is consistent with the estimate obtained by another method [32]†, $v_s \simeq 5.44/2\pi \times 3/2 \simeq 1.299$. The precision of the present estimate is comparable to the one in [32].

We also estimated the coefficient $B = v_s \pi \eta$ as

$$B = 1.0874 \pm 0.0002 . \tag{42}$$

This agrees with the previous estimate in [32], $B \simeq 0.725 \times \frac{3}{2} \simeq 1.088$.

Employing the formula $BC/A = \pi\eta$, we finally estimated the exponent η as

$$\eta = 0.269 \pm 0.001 . \tag{43}$$

This agrees with the conformal-field-theory prediction $\eta = \frac{4}{15} = 0.266\,666\ldots$.

Let us describe our data analysis in the following. We used the Lanczos method [33], and numerically diagonalized the systems of size up to $L = 16$.

The phenomenological-renormalization-group plot and the scaling plot are given in figure 5. The plots are completely explained by the values $\Gamma_c/J = 1$ and $\nu = \frac{5}{6}$, as expected.

† Note that the normalization factor of the Hamiltonian given in [32] is different from here. We have to multiply their Hamiltonian by $\frac{3}{2}$ to obtain our Hamiltonian.

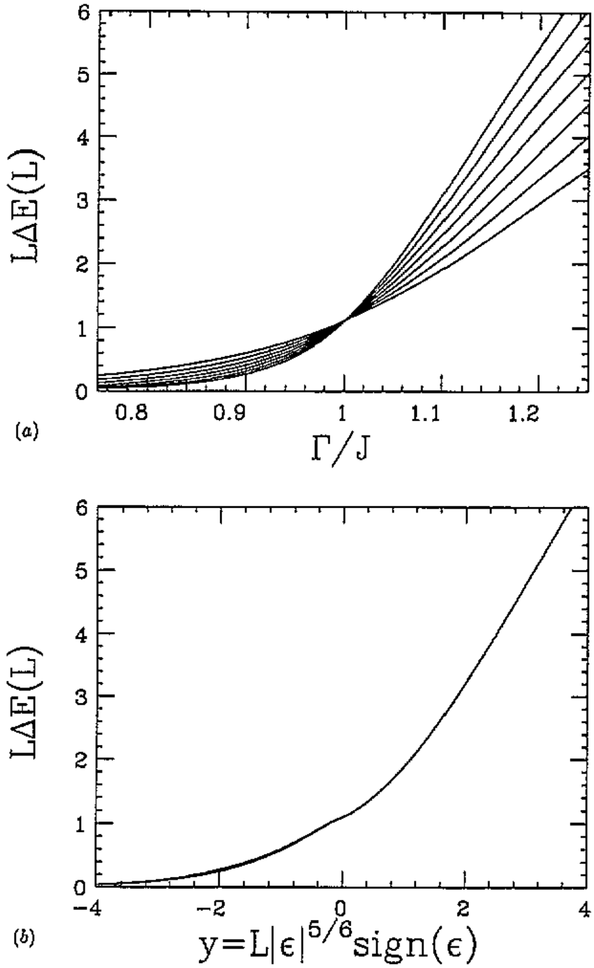

Figure 5. Scaling analysis for the quantum three-state Potts model: (*a*) the phenomenological-renormalization-group plot, $L\Delta E(L)$ versus Γ/J, and (b) the scaling plot, $L\Delta E(L)$ versus $y \equiv L|\varepsilon|^{5/6}$. The data for $7 \leqslant L \leqslant 13$ are shown here. The interval of the data points is $\Delta\Gamma = 0.05$.

First, we estimated the coefficient A in the disordered phase as follows. We plotted the quantity $A_L(y) \equiv L\Delta E(L)/y$ against y in the disordered phase. We expect from the scaling form (5) that the quantity $A_L(y)$ exponentially converges to the critical amplitude A in the limit $y \to \infty$. However, it is not the case as is exemplified in figure 6; the quantity $A_L(y)$ once reaches its minimum $A_L^{\min}$ at a certain point $y = y_L^{\min}$ ($\simeq 2.5$ in the present case), and begins to increase. This irregularity is probably because the scaling region is limited to $|y| < 2.5$. We hence evaluated the minimum $A_L^{\min}$ as an approximation of the coefficient A. The quantity $A_L^{\min}$ converges to the estimate (39) exponentially as $L \to \infty$; see figure 7. The error in (39) was evaluated by means of the least-squares fitting to a function of the form $A_L^{\min} = A - c_1 \exp(-c_2 L)$.

Next, we estimated the coefficient C in the ordered phase as follows. As was claimed below (6), we found that the linearity of the data $\log[L\Delta E(L)]$ against y is much better when we assume the behaviour $D_- \simeq \sqrt{y}$ of the prefactor in the scaling form (5). We

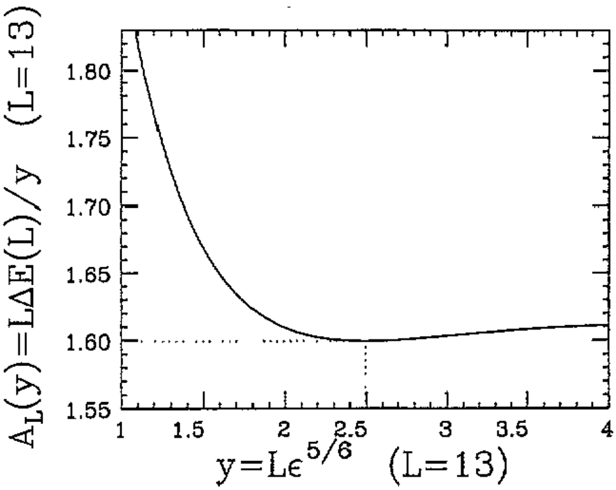

Figure 6. The quantity $A_L(y)$ versus y for $L = 13$. The minimum of $A_{13}(y)$ was estimated at $A_{13}^{\min} = 1.599\,426\,8$ for $y = y_{13}^{\min} = 2.494$.

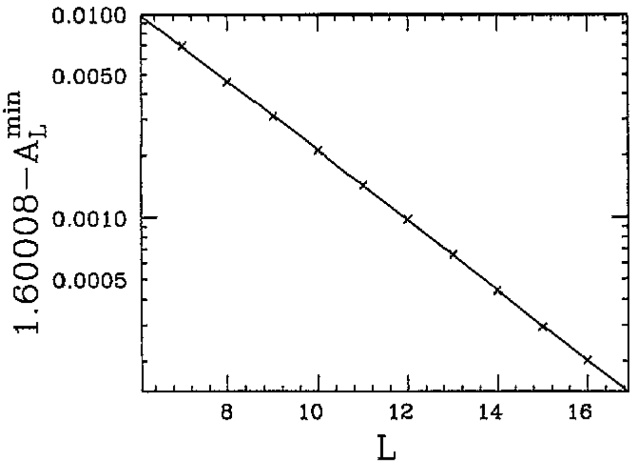

Figure 7. The exponential convergence of $A_L^{\min}$ to the estimate (39), $A = 1.600\,08$. The full line denotes the fitting line.

hence evaluated the following quantity in the ordered phase as an approximation of the coefficient C:

$$C_L \equiv -\left.\frac{\log\left[L\Delta E(L)/\sqrt{L|\varepsilon|^\nu}\right] - \log\left[(L-1)\Delta E(L-1)/\sqrt{(L-1)|\varepsilon|^\nu}\right]}{L|\varepsilon|^\nu - (L-1)|\varepsilon|^\nu}\right|_{y=y_L^{\min}} . \qquad (44)$$

(Here the point $y_L^{\min}$ is located in the ordered phase; we measured the coefficients A and C in the disordered phase and in the ordered phase, respectively, but at the same distance from the critical point.) The quantity converges to the estimate (40) exponentially as $L \to \infty$.

Finally, we estimated the coefficient B. The plot of the scaled energy gap $L\Delta E(L)$ at the critical point $\Gamma_c/J = 1$ manifests slight $1/L$ dependence of the data. The linear fitting to a function of the form $L\Delta E(L) = B + c_3/L$ yielded the estimate (42).

8. Summary

We have derived the amplitude relation $A/C = v_s$ both macroscopically and microscopically. We can thus introduce the universal amplitude ratio $BC/A = \pi\eta$.

Using the relation, we can estimate the exponent η directly. Only by means of the scaling analysis of energy-gap data can we obtain the critical point, the exponent ν and the exponent η. The relation has been confirmed analytically and numerically in three lattice models.

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